Space-time transformations within the path-integral approach to stochastic processes

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The use of space-time transformations within the path integral approach to quantum problems has made it possible to solve some ''complicated'' problems by their mapping into some simple, solvable, ones. Here we present a simple example on the possibility of exploiting this technique within the realm of stochastic processes, by analyzing the case of overdamped motion in a time-dependent harmonic potential. [S1063-651X(96)04406-6]

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I. INTRODUCTION

Since the pioneering work of Feynman and Hibbs [1], there have been numerous results related with basic aspects as well as applications of the path integral technique to different problems, but mainly in relation with quantum mechanics [2]. However, from a historical point of view, path integrals were first introduced in the thirties by Wiener to describe diffusion processes, and later used by Onsager and Machlup in the description of nonequilibrium and Markovian stochastic processes [3]. More recently, this technique has been extensively exploited to discuss several aspects related with stochastic and nonequilibrium processes and also extended to describe some non-Markovian stochastic processes [4].

One aspect recently studied by several authors concerns the application of space-time transformations, within the realm of path integral schemes, in order to ''map'' a ''difficult'' (in principle unsolvable) problem into a more simple (solvable) one [5]. One of the most outstanding examples is the possibility of solving the Coulomb problem, within a path integral framework, via the so called *Duru-Kleinert* transformations [5]. However the use of these transformations within the realm of the path integral approach to stochastic processes is scarce or almost inexistent [6].

In this paper we want to present a simple application of such a kind of transformations for the case of diffusion in a time-dependent harmonic potential. It is well known that a closed expression exists for the transition probability of such a system [7,8]. Also, a very thorough study of the most general Gaussian path integral form can be found in Ref. [9]. However, our aim is to show, through such a simple example, the possibility of exploiting these transformation techniques, within a path integral framework, in more complicated cases. We will follow the procedure presented in Felsager's book [10] for the quantum case, translating it to the stochastic (i.e., *imaginary time*) case. In what follows we present the procedure to be used, the way to get the *classical* (or *most probable*) trajectories necessary to write the general solution (that is reduced to quadratures), some examples, and a final discussion.

II. SPACE-TIME TRANSFORMATION

Our starting point is to consider the following Langevin equation:

$$\dot{x}(t)=h(x,t)+\xi(t), \tag{1}$$

where $\xi(t)$, as usual, is an additive *white noise* [7]. It is well known that in one dimension the multiplicative noise problem can be reduced to the additive one [7]. This corresponds to describing the overdamped motion of a particle in a time-dependent potential. In the present case we assume that $h(x,t)=-a(t)x$. As indicated, in Ref. [2(b)], the path integral representation of the transition probability associated with this Langevin equation is given by

$$p(x_b,t_b|x_a,t_a)=\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}[x(t)] \times\exp\left[-\int_{t_a}^{t_b}L(x(\tau),\dot{x}(\tau),\tau)d\tau\right]. \tag{2}$$

Here the stochastic *Lagrangian* or *Onsager-Machlup* functional [3] is given, in a midpoint discretization [2(b)], by

$$L(x,\dot{x},t)=\frac{1}{2D}[\dot{x}-h(x,t)]^2+\frac{1}{2}\frac{dh(x,t)}{dx}. \tag{3}$$

Replacing the actual form of $h(x,t)$, the previous expression can be expanded to yield:

$$L(x,\dot{x},t)=\frac{1}{2D}(\dot{x}^2+a^2x^2+2\,ax\dot{x})+\frac{1}{2}a=L_o+\frac{d\Phi(t)}{dt}, \tag{4}$$

$$\Phi(t)=-\frac{1}{2}\int_{t_a}^{t}a(t')dt'+\frac{a(t)}{2D}x^2, \tag{5}$$

$$L_o=\frac{1}{2D}[\dot{x}^2+[a^2(t)-\dot{a}(t)]x^2]. \tag{6}$$

At this point, as the Lagrangian of our problem is at the most quadratic in x and $\dot{x}$, we can obtain the exact result through the use of the usual procedure of expanding around a reference (*classical* or *most probable*) trajectory

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$[x(t)=x_{\mathrm{clas}}(t)+q(t)]$ and, besides the problem of getting such a trajectory, our problem reduces to evaluating the following expression:

$$p(x_b,t_b|x_a,t_a)=e^{-[\Phi(t_b)-\Phi(t_a)]}e^{-S_o^{\mathrm{clas}}(t_b,t_a)} \times\int_{q(t_a)=0}^{q(t_b)=0}\mathcal{D}[q(t)]e^{-\delta^2 S_o(t_b,t_a)}. \quad (7)$$

Here $S_o^{\mathrm{cl}}(t_b,t_a)$ is the action evaluated along the classical or most probable trajectory $x_{\mathrm{clas}(t)}$. The effective action to be solved in order to perform the path integral indicated in Eq. (7) is then:

$$\delta^2 S_o(t_b,t_a)=\frac{1}{2D}\int_{t_a}^{t_b}\left[\dot q^2+\left(a^2-\frac{da}{dt}\right)q^2\right]dt. \quad (8)$$

After partial integration, we get

$$\delta^2 S_o(t_b,t_a)=-\frac{1}{2D}\int_{t_a}^{t_b}\left[q\frac{d^2q}{dt^2}-\left(a^2-\frac{da}{dt}\right)q^2\right]. \quad (9)$$

Hence the path integral in Eq. (7) adopts the form:

$$p(x_b,t_b|x_a,t_a)=e^{-[\Phi(t_b)-\Phi(t_a)]}e^{-S_o^{\mathrm{clas}}(t_a,t_b)}\int_{q(t_a)=0}^{q(t_b)=0}\mathcal{D}[q(t)] \times\exp\left\{\frac{1}{2D}\int_{t_a}^{t_b}q\left[\frac{d^2}{dt^2}-\left(a^2-\frac{da}{dt}\right)\right]q\right\}. \quad (10)$$

In order to evaluate this path integral we need to diagonalize the operator

$$\frac{d^2}{dt^2}-\left[a^2-\frac{da}{dt}\right]. \quad (11)$$

However, there is an alternative way consisting in performing a change of variables that transforms the action in Eq. (8) into the one corresponding to the free diffusion problem. In following this approach we will make use of the results in Chap. 5 of Felsager's book [10]. Let us call $f(t)$ the solution of the equation

$$\left\{\frac{d^2}{dt^2}-w(t)\right\}f(t)=0, \quad (12)$$

where $w(t)=a^2-da/dt$, and with the condition $f(t_a)\neq 0$. It is easy to verify that

$$f(t)=A\exp\left\{-\int_{t_a}^{t}a(s)ds\right\} \quad (13)$$

is a solution of Eq. (12) fulfilling the required condition. We will now use the function $f(t)$ to perform the change of variables $q(t)\Rightarrow y(t)$ according to

$$q(t)=f(t)\int_{t_a}^{t}\frac{\dot y(s)}{f(s)}ds, \quad (14)$$

with the condition $y(t_a)=0$. Differentiating the previous expression we get

$$\dot q(t)=\dot f(t)\int_{t_a}^{t}\frac{\dot y(s)}{f(s)}ds+\dot y(t)=\frac{\dot f(t)}{f(t)}q(t)+\dot y(t) \quad (15)$$

that can be inverted yielding

$$y(t)=q(t)-\int_{t_a}^{t}\frac{\dot f(s)}{f(s)}q(s)ds. \quad (16)$$

Differentiating once more Eq. (15) we obtain

$$\ddot q(t)=\ddot f(t)\int_{t_a}^{t}\frac{\dot y(s)}{f(s)}ds+\frac{\dot f(t)\dot y(t)}{f(t)}+\ddot y(t). \quad (17)$$

Replacing the last result into the integrand of the exponent in Eq. (10), it can be transformed into

$$\left\{\frac{d^2}{dt^2}-w(t)\right\}q(t)=\{\ddot f(t)-w(t)f(t)\} \times\int_{t_a}^{t}\frac{\dot y(s)}{f(s)}ds+\frac{\dot f(t)\dot y(t)}{f(t)}+\ddot y(t). \quad (18)$$

Due to Eq. (12), the first term on the right-hand side (rhs) of Eq. (18) vanishes, reducing the effective action in Eq. (8) to

$$\delta^2 S_o[q(t)]=-\frac{1}{2D}\int_{t_a}^{t_b}dt\{F(t)\dot f(t)\dot y(t)+F(t)f(t)\ddot y(t)\}, \quad (19)$$

where

$$F(t)=\int_{t_a}^{t}ds\frac{\dot y(s)}{f(s)}. \quad (20)$$

The second term on the rhs of Eq. (19) can be integrated by parts leading to

$$\delta^2 S_o[q(t)]=\frac{1}{2D}\int_{t_a}^{t_b}dt\left(\frac{dy}{dt}\right)^2-\frac{1}{2D}[q(t)\dot y(t)]_{t_a}^{t_b}. \quad (21)$$

Due to the boundary conditions at $t=t_a$ and $t=t_b$, the second term of the last equation vanishes and we finally arrive at the action corresponding to free diffusion. The boundary conditions $q(t_a)=q(t_b)=0$, when written in terms of the new variable $y(t)$, have the form:

$$y(t_a)=0\,;\qquad \int_{t_a}^{t_b}ds\frac{\dot y(s)}{f(s)}=0. \quad (22)$$

However, the second boundary condition is a nonlocal one and therefore we shall resort to a special trick in order to handle it. Such a trick consists in using the identity $\delta[q(t_b)]=(2\pi)^{-1}\int_{-\infty}^{\infty}\exp[-isq(t_b)]ds$, in order to formally introduce the integration over the final endpoint:

$$\int_{q(t_a)=0}^{q(t_b)=0} D[q] e^{-\delta^2 S_o[q(t)]}$$
$$= \frac{1}{2\pi} \int_{q(t_a)=0}^{q(t_b)\text{arbitrary}} D[q] \int_{-\infty}^{\infty} ds e^{-isq(t_b)} e^{-\delta^2 S_o[q(t)]}, \quad (23)$$

where the integration over s produces the δ function that takes care of the correct boundary condition. Changing the integration variables $[q(t)\Rightarrow y(t)]$ we get

$$\frac{1}{2\pi} \int_{y(t_a)=0}^{y(t_b)\text{arbitrary}} D[y(t)] \int_{-\infty}^{\infty} ds$$
$$\times \exp\left(-is\left\{ f(t_b) \int_{t_a}^{t_b} [\dot{y}(s')/f(s')]\, ds' \right\} \right)$$
$$\times \exp\left[-(1/2D) \int_{t_a}^{t_b} dt (dy/dt)^2 \right] \det\left[\frac{\delta q}{\delta y} \right]. \quad (24)$$

As the transformation $q(t)\Rightarrow y(t)$ is linear, the Jacobian $\det[\delta q/\delta y]$ is independent of $y(t)$, and the remaining integral is Gaussian. ''Completing the square'' we get

$$= \frac{1}{2\pi} \det\left[\frac{\delta q}{\delta y} \right] \int_{-\infty}^{\infty} ds \, \exp\left\{ -(D/2)\, s^2 f^2(t_b) \int_{t_b}^{t_a} [dt/f^2(t)] \right\}$$
$$\times \int_{\vartheta(t_a)=0}^{\vartheta(t_b)\text{arbitrary}} D[\vartheta(t)] \exp\left[-\frac{1}{2D} \int_{t_a}^{t_b} dt \left(\frac{d\vartheta}{dt} \right)^2 \right], \quad (25)$$

with

$$\vartheta(t) = y(t) - iDsf(t_b) \int_{t_a}^{t} \frac{d\alpha}{f(\alpha)}. \quad (26)$$

The second integral in Eq. (25) is equal to unity as it represents the probability for finding the free diffusive system anywhere at time t_b. Hence, integrating the first one, we get the simple expression:

$$\int_{q(t_a)=0}^{q(t_b)=0} D[q] e^{-\delta^2 S_o[q(t)]} = \left[2\pi D f(t_a) f(t_b) \int_{t_a}^{t_b} \frac{dt}{f(t)^2} \right]^{-1/2}. \quad (27)$$

Here we have used that the Jacobian is given by $\det[\delta q/\delta y] = \sqrt{f(t_b)/f(t_a)}$ [10]. The final form for the transition probability is:

$$p(x_b, t_b | x_a, t_a) = e^{-[\Phi(t_b) - \Phi(t_a)]} e^{-S_o^{\text{clas}}(t_a, t_b)}$$
$$\times \left[2\pi D f(t_a) f(t_b) \int_{t_a}^{t_b} \frac{dt}{f(t)^2} \right]^{-1/2}. \quad (28)$$

Clearly, the expression for this transition probability contains the results for free diffusion $[\omega(t)=0,\ f(t)=1]$ and diffusion in a constant harmonic potential $\{\omega(t)=\omega^2 = ct.,\ f(t)=\cosh[\omega(t-t_a)]\}$. In the general case, the function $f(t)$is given by Eq. (13). It is worth remarking that the result in Eq. (28) is in complete agreement with those obtained by previous authors by other means (see for instance [9]).

In order to completely solve the problem, i.e., to have the transition probability in Eq. (28), we shall evaluate $S_o^{\text{clas}}(t_b, t_a) = S_o[x_{\text{clas}}(t)]$, where $x_{\text{clas}}(t)$ is the solution of the Euler-Lagrange equation

$$\left[\frac{d^2}{dt^2} - \left(a^2 - \frac{da}{dt} \right) \right] x_{\text{clas}}(t) = 0 \quad (29)$$

fulfilling the ''boundary conditions'': $x_{\text{clas}}(t_a) = x_a$ and $x_{\text{clas}}(t_b) = x_b$. The general form of such a solution is

$$x_{\text{clas}}(t) = B \, \exp\left[\left\{ -\int_{t_a}^{t} a(s) ds \right\} \right] + g(t), \quad (30)$$

where $g(t)$ is an independent solution of the equation of motion that shall be obtained for each $a(t)$. It is easy to check that

$$g(t) = \exp\left(\left\{ -\int_{t_a}^{t} a(s) ds \right\} \right) \int_{t_a}^{t} d\tau \, \exp\left(\left\{ 2 \int_{t_a}^{\tau} a(z) dz \right\} \right) \quad (31)$$

is a convenient form of the desired independent solution. The knowledge of $x_{\text{clas}}(t)$ allows us to evaluate S_o^{clas}. Hence, we have reached an expression where, given the form of the time dependence of $a(t)$, the complete solution of the problem is reduced to *quadratures*. In the next section we will exploit this general result for a couple of analytically solvable examples.

III. ANALYTICAL SOLUTIONS

We propose here a simple method to generate a whole family of analytical classical solutions. In order to reach this goal we write the elastic parameter in the following form:

$$a(t) = b(t) + \frac{1}{2} \frac{b'(t)}{b(t)}. \quad (32)$$

This allows us to find the forms

$$x_1(t) = \frac{\sinh[B(t)]}{\sqrt{b(t)}},$$
$$x_2(t) = \frac{\cosh[B(t)]}{\sqrt{b(t)}} \quad (33)$$

for the classical solutions. The only condition in order to get analytical trajectories is that $b(t)$ must be an integrable function as we have that $B(t) = \int_{t_0}^{t} b(s) ds$. With this choice of $a(t)$, the solution of Eq. (29), with boundary conditions $x_{\text{clas}}(t_a) = x_a$ and $x_{\text{clas}}(t_b) = x_b$, is given by

$$x_{\text{clas}} = \frac{x_a \sqrt{b(t_a)} \sinh[B(t_b)] - x_b \sqrt{b(t_b)} \sinh[B(t_a)]}{\sinh[B(t_b) - B(t_a)]}$$
$$\times \frac{\cosh[B(t)]}{\sqrt{b(t)}}$$
$$+ \frac{x_b \sqrt{b(t_b)} \cosh[B(t_a)] - x_a \sqrt{b(t_a)} \cosh[B(t_b)]}{\sinh[B(t_b) - B(t_a)]}$$
$$\times \frac{\sinh[B(t)]}{\sqrt{b(t)}}. \quad (34)$$

With these results, the function $\Phi(t)$ [Eq. (5)] becomes

$$\Phi(t)=-\frac{B(t)}{2}-\frac{1}{4}\ln[b(t)]+\frac{a(t)}{2D}x^2(t). \qquad (35)$$

Hence the final expresion for the transition probability is

$$p(x_b,t_b|x_a,t_a) = \sqrt{\frac{\sqrt{b(t_a)b(t_b)}}{4\pi D\sinh[B(t_b)-B(t_a)]}}\left(\frac{b(t_b)}{b(t_a)}\right)^{1/4} \times\exp\left[\frac{B(t_b)-B(t_a)}{2}\right] \times\exp\left[\frac{1}{2D}[a(t_b)x_b^2-a(t_a)x_a^2]\right] \times\exp\left[\frac{-1}{2D}[x_b\dot{x}(t_b)-x_a\dot{x}(t_a)]\right]. \qquad (36)$$

Here $\dot{x}(t_a)$ and $\dot{x}(t_b)$ are the time derivative of x_{clas}(t) at t_a and t_b, respectively.

A. Examples

It is clear that there are a large number of possibilities of functional forms for $b(t)$ leading to analytical results. Here, we start considering the following form for the function $b(t)$:

$$b(t)=\frac{\gamma}{t}$$

that yields for $a(t)$ the form

$$a(t)=\frac{\alpha}{t},$$

with $\alpha=\gamma-\frac{1}{2}$. This choice corresponds to a system that goes asymptotically to free diffusion. With this choice we can obtain the solution for a whole family of functions corresponding to different values of γ. Replacing this form into the previous expressions yields for the solutions $x_1(t)$ and $x_2(t)$:

$$x_1(t)\simeq t^{-\alpha}, \quad x_2(t)\simeq t^{1+\alpha},$$

while for the transition probability we find

$$p(x_b,t_b|x_a,t_a)=\left\{\frac{4\pi D\sqrt{t_at_b}}{2\gamma}\left[\left(\frac{t_b}{t_a}\right)^{\gamma}-\left(\frac{t_a}{t_b}\right)^{\gamma}\right]\right\}^{-1/2}\left(\frac{t_b}{t_a}\right)^{(\alpha/2)}\exp\left[-\frac{\alpha}{2D}\left(\frac{x_b^2}{t_b}-\frac{x_a^2}{t_a}\right)\right] \times\exp\left[-\frac{1}{2D}\left(\frac{2(2\alpha+1)x_ax_bt_a^{\alpha}t_b^{\alpha}-x_b^2[(\alpha+1)t_b^{2\alpha}+t_a^{2\alpha+1}/t_b]-x_a^2[(\alpha+1)t_a^{2\alpha}+t_b^{2\alpha+1}/t_a]}{t_a^{2\alpha+1}-t_b^{2\alpha+1}}\right)\right] \qquad (37)$$

Clearly, when $\gamma=0$ (correspondingly $\alpha=-\frac{1}{2}$) we meet a kind of singular situation. This corresponds to the coalescence of two ''classical trajectories,'' and its solution requires a special treatment as we cannot exploit the previous forms for the classical solution. However, we can overcome this difficulty writing the new solutions as

$$x_1(t)=\sqrt{t}, \qquad (38)$$

$$x_2(t)=\sqrt{t}\ln[t].$$

Using these forms into the previous expressions, leads us to the following form of the propagator

$$p(x_b,t_b|x_a,t_a)=\left[2\pi D\sqrt{t_at_b}\ln\left(\frac{t_b}{t_a}\right)\right]^{(-1/2)}\left(\frac{t_b}{t_a}\right)^{(-1/4)}\exp\left[\frac{1}{4D}\left(\frac{x_b^2}{t_b}-\frac{x_a^2}{t_a}\right)\right] \times\exp\left[-\frac{1}{2D}\left(\frac{\frac{x_a}{\sqrt{t_a}}\left(1+\frac{\ln[t_b]}{2}\right)-\frac{x_b}{\sqrt{t_b}}\left(1+\frac{\ln[t_a]}{2}\right)}{\ln[t_b]-\ln[t_a]}\right)\left(\frac{x_b}{\sqrt{t_b}}-\frac{x_a}{\sqrt{t_a}}\right)\right] \times\exp\left[-\frac{1}{2D}\left(\frac{\frac{x_a}{\sqrt{t_a}}-\frac{x_b}{\sqrt{t_b}}}{\ln[t_b]-\ln[t_a]}\right)\left(\frac{x_b\ln[t_b]}{\sqrt{t_b}}-\frac{x_a\ln[t_a]}{\sqrt{t_a}}\right)\right]. \qquad (39)$$

This corresponds to a limit case of the previous propagator that, however, cannot be obtained from Eq. (37) in a trivial way.

Here, and for the sake of completeness, we show a couple of final examples for $b(t)$ that could be of some interest, but without elaborating on the final form of the propagator.

As the first case, we consider a case that can be reduced to an oscillator with a frequency that oscillates around a prescribed value:

$$b(t)=\mathrm{K}+\alpha\ \sin\omega t \tag{40}$$

$$a(t)=\mathrm{K}+\alpha\sin\omega t+\frac{1}{2}\frac{\alpha\omega\ \cos\omega t}{\mathrm{K}+\alpha\ \sin\omega t}.$$

In the limit of $\alpha\ll K$ we reduce to the above indicated case with $a(t)\approx\mathrm{K}+\alpha\ \sin\omega t$.

A second interesting situation is the case when the frequency changes from a given initial value (at $t=0$) to another fixed value (for $t\to\infty$). We can propose:

$$b(t)=\mathrm{K}+\alpha\mathrm{e}^{-t/\tau} \tag{41}$$

$$a(t)=\mathrm{K}+\alpha\mathrm{e}^{-t/\tau}\left[1-\frac{1}{2\tau(\mathrm{K}+\alpha\mathrm{e}^{-t/\tau})}\right].$$

We see that the limit values correspond to $a(0)=\mathrm{K}+\alpha\{1-[1/2\tau(\mathrm{K}+\alpha)]\}$ and $a(\infty)=\mathrm{K}$, respectively.

It is clear that, given the forms of $b(t)$ and $a(t)$, it is simply to get $B(t)$, and replacing all these functions into Eqs. (34)–(36), we can obtain closed expresions for the propagators.

IV. CONCLUSIONS

In this paper we have addressed, through a simple case, the problem of exploiting space-time transformations [5] within the realm of the path integral approach to stochastic processes. We have considered as a simple application of those transformations the problem of diffusion in a time-dependent harmonic potential. This problem has been studied, using standard techniques, by other authors [9]. In order to proceed with the calculation we have profited from the results for the quantum case as presented in Felsager's book [10], adapting it to the stochastic case.

We have shown the procedure to be used, how to get the *classical* or *most probable* trajectory (needed to write the general solution) and also shown that in this particular case the general solution is reduced to quadratures. We have presented the solution for a whole family of analytical solutions when we write the elastic parameter in a particular form, and have presented a few examples corresponding to (repulsive or attractive) potentials that asymptotically go over the free diffusion case. We have particularized the case of coalescence of classical trajectories. Finally, we have included a couple of other interesting examples. In most of these cases we have obtained the expression for the final form of the transition probability or propagator. All these examples indicate some of the possibilities of this approach. What still remains open is the analysis of the same problem without exploiting the connection between the Fokker-Planck and Schrödinger equations, but working in the original non-Hermitian framework. Clearly, this will be a necessary step when studying higher dimensional systems. As interesting applications of the present results we can indicate some time-dependent problems such as those discussed in Refs. [11–13].

However, the main interest will be to apply, adapt, or extend, the more general form of space-time transformations [5], i.e., of the Duru-Kleinert type, within the path integral approach to stochastic processes, in order to ''map'' a ''difficult'' (in principle unsolvable) problem into a more simple (solvable) one. The study of the many aspects of this problem will be the subject of further work.

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